Vortices in multicomponent exciton-polariton superfluids

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Outline:

- XY-model. BKT transition. Superfluids
- Spin-1 condensates. The Berry phase. Half-vortices
- Spin-2 condensates. One-third vortices
- Condensation of exciton-polaritons in microcavities
- Half-vortices in exciton-polariton condensates
- Warping of vortices in the presence of TE-TM splitting
- Interactions and peculiarities of BKT transition
- Observations of half-vortices and polarization vortices
Phase-transition in XY-model

The symmetry is lowered due to the disorder-order transition:

\[
T > T_c \\
T < T_c
\]

\[
H = \rho_s \sum_{(ij)} [1 - \cos(\theta_i - \theta_j)] = \frac{1}{2} \rho_s \sum_{(ij)} [\theta(\vec{r}_i) - \theta(\vec{r}_j)]^2
\]

\[
= \frac{1}{2} \rho_s \sum_i a^2 |\nabla \theta(\vec{r}_i)|^2 = \frac{1}{2} \rho_s \int d^2 r |\nabla \theta(\vec{r})|^2.
\]
Apart from spin waves, the vortices are important (V. L. Berezinskii, 1970). In vortex, there is rotation of the spin on distances $\gg a$ (the core size), with a resulting change multiple by $2\pi$:

$$\text{Core}$$

Winding number $= 1$

$$\text{Core}$$

Winding number $= 2$

Variation of $\theta(\vec{r})$ is subject to minimization of the Hamiltonian energy,

$$-\rho_s \Delta \theta(\vec{r}) = 0, \quad \theta = \theta_0 + n_w \phi.$$

The winding number $n_w = 0, \pm 1, \pm 2, \ldots$. Mathematically, $\pi_1(S_1) = \mathbb{Z}$. 

XY-model: Vortices and winding numbers
Topological invariance and topological stability

\[ n_w = n_1 + n_2 + n_3 \]

Single vortex is topologically stable. It cannot be transformed into the ground state \((n_w = 0)\).

A \(n_1 = +1\) and \(n_2 = -1\) vortex pair can be created from and transformed into the ground state, since \(n_1 + n_2 = 0\).
The Berezinskii-Kosterlitz-Thouless (BKT) transition

The single vortex energy

\[ E_s = \frac{1}{2} \rho_s \int d^2 r |\nabla \theta|^2 = \frac{1}{2} \rho_s \int d^2 r \left( \frac{1}{r} \frac{d}{d\phi} \right)^2 = \pi \rho_s k^2 \int_a^R \frac{1}{r} dr = \pi \rho_s n_{12}^2 \ln \left( \frac{R}{a} \right). \]

The energy of a pair of vortices is finite for \( n_1 + n_2 = 0 \),

\[ E_p = \pi \rho_s (n_1 + n_2)^2 \ln(R/a) - 2\pi \rho_s n_1 n_2 \ln(r/a). \]

The critical temperature can be found by

\[ F = E_s - TS = \pi \rho_s \ln(R/a) - T \ln(R^2/a^2) = (\pi \rho_s - 2T) \ln(R/a), \]

so that single vortices appear and destroy the order at

\[ T_c = \frac{\pi}{2} \rho_s. \]
Bose-Einstein condensation (BEC)

Ideal Bose gas. In the 3D case:

\[ N = N_0 + \frac{V}{(2\pi)^3} \int_0^\infty \frac{4\pi k^2 dk}{\exp\left\{ \frac{\omega(k) - \mu}{T} \right\} - 1} = N_0 + N_{\text{ex}}(\mu, T), \quad \omega_0(k) = \frac{k^2}{2m}. \]

Condensation temperature \( T_{\text{BE}} \) is the root of \( N_{\text{ex}}(0, T_{\text{BE}}) = N \), so that \( \mu < 0 \) for \( T > T_{\text{BE}} \). One can omit \( N_0 \).

\( \mu = 0 \) for \( T < T_{\text{BE}} \) and \( N_0 \propto V \).

There is macroscopic number of bosons in the state with \( \vec{k} = 0 \).

The order parameter \( \psi = \sqrt{n} e^{i\phi} \), where \( n = N_0/V \).

The result is different in the 2D case:

\[ N_{\text{ex}}(\mu, T) = \frac{A}{(2\pi)^2} \int_0^\infty \frac{2\pi kdk}{\exp\left\{ \frac{\omega(k) - \mu}{T} \right\} - 1} \]

is divergent at \( \mu = 0, T > 0 \).

so that \( \mu < 0 \) for all \( T > 0 \).

There is no BEC (as a phase transition) in 2D at finite temperatures.

This is a particular case of the Mermin-Wagner theorem about the absence of true long-range order in 2D at \( T > 0 \).
Effects of interactions. Superfluidity

Repulsive interaction $U_0 \equiv U(k = 0) > 0$ changes qualitatively the quasiparticle spectrum. One has $\mu = nU_0 > 0$ and (N. N. Bogoliubov, 1947):

$$\omega^2(k) = \omega_0^2(k) + 2nU_0\omega_0(k) = \omega_0^2(k) + 2\mu\omega_0(k), \quad \omega_0 = k^2/2m^*,$$

so that $\omega = v_0k$ with the sound velocity $v_0 = \sqrt{nU_0/m^*}$ for small $k$.

This results in superfluidity (L. D. Landau, 1940). One needs

$$Mv = P = P' + k,$$

$$\frac{p^2}{2M} = \frac{(P-k)^2}{2M} + \omega(k), \text{ or } \omega(k) = kv.$$

The flow with $v < \omega(k)/k$ is ideal. No losses and viscosity.

In 2D case, a weak repulsive interaction makes possible a superfluid transition with $T_c > 0$. 
Mean-field theory

\[ H = \int d^d r \left\{ \frac{\hbar^2}{2m^*} | \nabla \psi(r) | ^2 - \mu | \psi | ^2 + \frac{1}{2} U_0 | \psi | ^4 \right\} . \]

\[ T \to 0 : \mu = U_0 n, \quad \psi = \sqrt{n_c} e^{i \theta} \]

Considering vortices one can assume \(| \psi | ^2 = n_c\) (otherwise energy \( \propto \) area), so \( \psi = \sqrt{n_c} e^{i \theta(r)} \), and \( H_{el} = \frac{1}{2} \rho_s \int d^d r | \nabla \theta(r) | ^2 \), \( \rho_s = \frac{\hbar^2 n_c}{2m^*} \).

**The vortex core.** Condition \(| \psi | = \sqrt{n_c}|\) is violated on distances \( r \lesssim a = \hbar/\sqrt{2m^*\mu} \), that define the vortex core.

Singularity at \( r \to 0 \), where \( \psi \propto r e^{\pm i \phi} \), \( r^2 e^{\pm 2i \phi} \), \( \ldots \) for \( n_w = \pm 1, \pm 2, \ldots \).
Multicomponent superfluids. Mixing of the components

The order parameter \( \psi = (\psi_F, \psi_{F-1}, \ldots, \psi_m, \ldots, \psi_{-F})^T \).

Simple classification: vortex in one component, with the other component being regular. If \( \psi_i = |\psi_i|e^{i\theta_i} \), one can have

\[ \theta_m \rightarrow \theta_m + n_w \phi, \quad \theta_{m'} \rightarrow \theta_{m'} \quad (\text{for } m' \neq m). \]

In general, this classification is not good: there is mixing of components, and the phases are not independent.

Mixing due scattering. E.g., of two \( m = 0 \) atoms into \( m = +1 \) and \( m = -1 \):

\[
H_{\text{mix}} = \frac{1}{2} V_{\text{mix}} (\psi^*_+ \psi^-_0 \psi_0 \psi_0 + \psi^*_0 \psi^-_1 \psi^*_1 \psi^-_1)
= V_{\text{mix}} |\psi^*_+| |\psi^-_0| |\psi_0|^2 \cos(\theta^*_+ + \theta^-_1 - 2\theta_0).
\]

So that the change of phases is subject to the constriction \( \theta^*_+ + \theta^-_1 - 2\theta_0 = n\pi \).

The other source of mixing is possible dependence of the mass of the particle on the direction of \( \psi \) (like longitudinal-transverse splitting).
Spin=1 BEC: Ferromagnetic and Polar Phases

For \( \psi = (\psi_+, \psi_0, \psi_-)^T \) the interaction energy is

\[
H_{\text{int}} \propto V_0 n^2 + V_1 |\vec{F}|^2, \quad n = \psi^\dagger \cdot \psi, \quad \vec{F} = \psi^\dagger \cdot \vec{f} \cdot \psi,
\]

\[
f_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Rotation operator \( U(\alpha, \beta, \gamma) = e^{-if_z \alpha} e^{-if_y \beta} e^{-if_z \gamma} \).

<table>
<thead>
<tr>
<th>Ferromagnetic phase ( V_1 &lt; 0 )</th>
<th>Polar phase ( V_1 &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi = \sqrt{n} e^{i\theta} U(\alpha, \beta, 0) \begin{pmatrix} 1 \ 0 \ 0 \end{pmatrix} )</td>
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<tr>
<td>( = \sqrt{n} e^{i\theta} \begin{pmatrix} \sqrt{2} e^{-i\alpha} \cos(\beta/2) \ \sin(\beta) \ \sqrt{2} e^{i\alpha} \sin(\beta/2) \end{pmatrix} )</td>
<td>( = \sqrt{n} e^{i\theta} \begin{pmatrix} -e^{-i\alpha} \sin(\beta) \ \sqrt{2} \cos(\beta) \ e^{i\alpha} \sin(\beta) \end{pmatrix} ).</td>
</tr>
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Spin=1 Ferromagnetic: The Berry phase

\[ \hat{s} = \frac{\mathbf{F}}{n} = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta) \]

The mass velocity

\[ \mathbf{v} = \frac{i\hbar}{2m^* n} \left( \psi_m (\vec{\nabla} \psi^*_m) - \psi^*_m (\vec{\nabla} \psi_m) \right) = \frac{\hbar}{m^*} (\vec{\nabla} \theta - \cos \beta \vec{\nabla} \alpha) \]

The Mermin-Ho relation:

\[ \vec{\nabla} \times \mathbf{v} = (\hbar/2m^*) \sin \beta (\vec{\nabla} \beta \times \vec{\nabla} \alpha) = (\hbar/2m^*) \epsilon_{\mu \nu \lambda} \delta_{\mu} [\vec{\nabla} \delta_{\nu} \times \vec{\nabla} \delta_{\lambda}] \neq 0. \]

\[ \oint \mathbf{v} \cdot d\mathbf{l} = \frac{\hbar}{m^*} \oint (1 - \cos \beta) \vec{\nabla} \alpha \cdot d\mathbf{l} = \frac{\hbar}{m^*} n_w, \text{ with } n_w \in \mathbb{Z}. \]
Spin=1 Ferromagnetic: Topological stability and vortex creation.

Consider $\alpha = n_\omega \phi$ and $\theta = -n_\omega \phi$:

$$\psi = \sqrt{n} \left( \frac{\cos^2 \beta}{2}, \frac{\sin \beta}{\sqrt{2}}, e^{\text{i} n_\omega \phi} \frac{\sin^2 \beta}{2} \right)^T = \sqrt{n} \left\{ \begin{array}{l} (1, 0, 0) \text{ for } \beta = 0, \\ (0, 0, e^{2\text{i} n_\omega \phi}) \text{ for } \beta = \pi. \end{array} \right.$$  

The vortex in $\pm 1$-component with an even winding number is unstable. It can be used to create such a vortex form the ground state by applying adiabatic magnetic field (W. Ketterle group, 2002):

$$\vec{B} = (B_\perp \cos(-\phi), B_\perp \sin(-\phi), B_z)^T,$$

that gives $\alpha = \phi$ and $\beta = \arctan(B_\perp / B_z)$.

Also,

$$(0, 0, e^{\text{i}(2n_\omega + 1)\phi})^T \leftrightarrow (e^{\text{i} \phi}, 0, 0)^T.$$
Spin=1 Ferromagnetic: Two types of vortices

The coreless vortex. \( \theta = \alpha = \pm \phi, \quad \beta(r = 0) = 0, \quad \beta(r = r_0) = \pi, \)

\[
\psi_{cl} = \sqrt{n} \begin{pmatrix}
\cos^2(\beta/2) \\
e^{i\phi} \sin \beta \\
e^{2i\phi} \sin^2(\beta/2)
\end{pmatrix}, \quad \vec{v} = \frac{\hbar}{m^* r} (1 - \cos \beta) \hat{\phi}.
\]

The polar-core vortex. \( \theta = 0, \quad \alpha = \pm \phi, \quad \beta = \text{const}, \)

\[
\psi_{pl} = \sqrt{n} \begin{pmatrix}
e^{-i\phi} f(r) \cos^2(\beta/2) \\
[1 - f^2(r) \cos^4(\beta/2) - g^2(r) \sin^4(\beta/2)]^{1/2} \\
e^{i\phi} g(r) \sin^2(\beta/2)
\end{pmatrix},
\]

where \( f(0) = g(0) = 0, \quad f(\infty) = g(\infty) = 1. \)
Spin=1 Ferromagnetic: Two types of vortices

Profile of the order parameter: $\sum_m \psi_m Y_{1m}(\delta)$

(a)  
(b)  
(c)  

[From Ueda and Kawaguchi, (2010)]
Spin=1 Polar: Half-vortices

Spin quantization axis \( \hat{d} = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta) \)

\[
\psi_{\text{pol}} = \sqrt{\frac{n}{2}} e^{i\theta} \begin{pmatrix} -e^{-i\alpha} \sin \beta \\ \sqrt{2} \cos \beta \\ e^{i\alpha} \sin \beta \end{pmatrix}, \quad (\hat{d} \cdot \hat{f})\psi_{\text{pol}} = 0, \quad \vec{v} = \frac{\hbar}{m^*} \vec{\nabla} \theta.
\]

While there is no Berry phase in quantization of supercurrent, the spin-gauge symmetry leads to existence of half-quantum vortices:

\[ \psi \to \psi, \quad \text{when} \quad \hat{d} \to -\hat{d}, \quad \theta \to \theta \pm \pi. \]

For example, setting \( \beta = \pi/2, \alpha = n_w \phi/2, \theta = n_w \phi/2, \) we have

\[
\psi = \sqrt{\frac{n}{2}} \begin{pmatrix} -1 \\ 0 \\ e^{in_w \phi} \end{pmatrix}, \quad \int \vec{v} \cdot d\vec{l} = \frac{\hbar}{2m^*} n_w.
\]

In 3D the half-vortex line is also referred to as the Alice string.
Spin=1 Polar: Half-vortices

The order parameter defines the surface $|\sum_m \psi_m Y_{1m}(\hat{s})|^2$. Colors indicate combined spin-gauge symmetry.

[From Ueda and Kawaguchi, (2010)]
Phases of Spin=2 Condensates

\[ H_{\text{int}} \propto V_0 n^2 + V_1 |\vec{F}|^2 + V_2 |A|^2, \quad A = [2\psi_-\psi_2 - 2\psi_1\psi_1 + \psi_0^2]/\sqrt{5}. \]
One-third vortices in the cyclic phase

Cyclic state \( \xi_{\text{cyc}} = (1, 0, i\sqrt{2}, 0, 1)^T \). For \( \varphi = 2\pi/3 \)

\[
\exp \left\{ -i \frac{f_x + f_y + f_z}{\sqrt{3}} \varphi \right\} e^{i\varphi} \xi_{\text{cyc}} = e^{4\pi i/3} e^{i\varphi} \xi_{\text{cyc}}, \quad \theta \to \theta + \frac{2\pi}{3}.
\]

[From Ueda and Kawaguchi, (2010)]
References


- Effective mass of lower branch polaritons: $m^* \sim 10^{-4}m_0$.
- Relaxation problem: bottleneck for $\frac{d\omega}{dk} > v_{\text{sound}}$.
- Presence of polarization degree of freedom. The order parameter is complex 2D vector $\tilde{\psi} \propto \tilde{E}_{||}$.
First observation of condensation (Le Si Dang group)

Emission from the CdTe-based microcavity:

Spontaneous polarization formation (Le Si Dang group)

Linear polarization of the condensate

Formation of linear polarization in polariton condensates [Le Si Dang et al.; Snoke et al., 2006] arises due to the reduction of polariton-polariton repulsion energy $H_{\text{int}}$:

$$H_{\text{int}} = \frac{1}{2} \int d^2r \left\{ (U_0 - U_1)(\tilde{\psi}^* \cdot \tilde{\psi})^2 + U_1 |\tilde{\psi}^* \times \tilde{\psi}|^2 \right\}.$$

Two interaction constants, $U_0 = AM_{\uparrow\downarrow}$ and $U_1 = A(M_{\uparrow\downarrow} - M_{\uparrow\uparrow})/2$, where $A = \pi R^2$ is the excitation spot area. Typically, $U_0/2 < U_1 < U_0$.

At a fixed concentration $n = (\tilde{\psi}^* \cdot \tilde{\psi})$ minimum of $H_{\text{int}}$ is reached for $\tilde{\psi}^* \times \tilde{\psi} = 0 \Rightarrow$ Linear polarization

One can write $\tilde{\psi}_{\text{lin}} = \{\psi_x, \psi_y\} = \sqrt{n} e^{i\varphi} \{\cos \eta, \sin \eta\}$,

so that the order parameter is defined by two angles, $\eta$ and $\theta$.

The states $\eta, \theta$ and $\eta + \pi, \theta + \pi$ are identical.

The order parameter manifold $M = (U(1) \times S_1)/\mathbb{Z}_2$.

The first homotopy group $\pi_1(M) = \mathbb{Z} \times \mathbb{Z}$. 
The order parameter space

$$\tilde{\psi}_{\text{lin}} = \{\psi_x, \psi_y\} = \sqrt{n} e^{i\theta} \{\cos \eta, \sin \eta\}.$$  

The possible changes are:

$$\eta \rightarrow \eta + 2\pi k,$$
$$\theta \rightarrow \theta + 2\pi m.$$  

Vortex carries two topological charges (winding numbers), $$(k, m).$$

Integer vortices:

$$k, m = 0, \pm 1, \pm 2, \ldots$$

Half-integer vortices:

$$k, m = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots$$
Half-vortices


They appear due to combined spin-gauge symmetry:

Spin quantization axis change $\vec{d} \rightarrow -\vec{d}$
Phase change $\theta \rightarrow \theta \pm \pi$

The superfluid velocity around the half-vortex $\vec{u}_s \propto \nabla \theta$ is a half of the superfluid velocity around the usual vortex with $\theta \rightarrow \theta \pm 2\pi$.

Half-vortex carries half-quantum of the superfluid current.
Why two winding numbers \((k, m)\)?

Atomic spin-1 condensates (three-component)

- 3D real \(\vec{d}\) and phase \(\theta\)
- Half-vortex: \(\vec{d} \rightarrow -\vec{d}, \theta \rightarrow \theta + \pi\)

All rotations \(\vec{d} \rightarrow -\vec{d}\) can be transformed to each other

Polariton pseudospin case (two-component)

- 2D real \(\vec{d}\) and phase \(\theta\)
- Half-vortex: \(\vec{d} \rightarrow -\vec{d}, \theta \rightarrow \theta + \pi\)

Clockwise and counterclockwise \(\vec{d} \rightarrow -\vec{d}\) are topologically distinct
Half-vortex in the circular polarization basis

The circular components are defined by

$$\psi = \hat{x}\psi_x + \hat{y}\psi_y = \frac{\hat{x} + i\hat{y}}{\sqrt{2}}\psi_{+1} + \frac{\hat{x} - i\hat{y}}{\sqrt{2}}\psi_{-1}. $$

Consider $\eta = k\phi$ and $\theta = m\phi$, where $\phi$ is the azimuthal angle, then

$$\psi_{+1} = \sqrt{\frac{n}{2}} e^{i(m-k)\phi}, \quad \psi_{-1} = \sqrt{\frac{n}{2}} e^{i(m+k)\phi}. $$

Right half-vortices: $k = m$. Left-circular component becomes fully depleted and polarization is right-circular at $r = 0$.

Left half-vortices: $k = -m$. Right-circular component becomes fully depleted and polarization is left-circular at $r = 0$. 
Integer vortices

(0, +1) \rightarrow (+\frac{1}{2}, +\frac{1}{2}) + (\frac{1}{2}, +\frac{1}{2}),
(-1, 0) \rightarrow (-\frac{1}{2}, +\frac{1}{2}) + (-\frac{1}{2}, -\frac{1}{2}),
(+1, 0) \rightarrow (+\frac{1}{2}, +\frac{1}{2}) + (+\frac{1}{2}, +\frac{1}{2}).

The hedgehog (+1, 0) restores the polarization $S_1$ symmetry ($C_{\infty}$).
The elastic energy for $m_t = m_\perp = m^*$

The elastic energy in the case when polariton mass does not depend on polarization

$$H_{el} = \frac{1}{2} \rho_s \int d^2 r \left[ (\nabla \eta)^2 + (\nabla \theta)^2 \right], \quad \rho_s = \frac{\hbar^2 n}{m^*}.$$  

The elastic field of half-vortices is simple: $\Delta \theta = 0$ and $\Delta \eta = 0$, so that $\theta = \pm \frac{1}{2} \phi$ and $\eta = \pm \frac{1}{2} \phi$. Logarithmic prefactor and $T_{KT}$:

$$E_s = \frac{1}{2} \pi \rho_s, \quad T_{KT} = \frac{1}{2} E_s = \frac{1}{4} \pi \rho_s.$$  

The other way is to introduce the phases of circular components, $\theta_+ = \theta - \eta$ and $\theta_- = \theta + \eta$, then

$$H_{el} = \frac{1}{4} \rho_s \int d^2 r \left[ (\nabla \theta_+)^2 + (\nabla \theta_-)^2 \right],$$

and the half-vortex is a full vortex in only one circular component.
The polarization texture of half-vortex core

Showing $\text{Re}\{\psi e^{-i\omega t}\}$, where $\omega = \omega_p + \mu$.

“Lemon”

“Star”
Two left half-vortices
Pair of left half-vortices
Permuted pair of left half-vortices
The Gross-Pitaevskii equation

The Gross-Pitaevskii equation is

$$i\hbar \frac{\partial}{\partial t} \tilde{\psi}(\vec{r}, t) = \frac{\delta H}{\delta \tilde{\psi}^*(\vec{r}, t)}, \quad H = \int \mathcal{H}(\tilde{\psi}^*, \tilde{\psi}) d^2 r,$$

The energy density for the polariton superfluid:

$$\mathcal{H} = \mathcal{T} - \mu n + \mathcal{H}_{\text{int}} + \mathcal{H}'.$$

Spin-dependence of the kinetic energy and interactions

$$\mathcal{T} = \frac{\hbar^2}{2m_l} |\nabla \cdot \tilde{\psi}|^2 + \frac{\hbar^2}{2m_t} |\nabla \times \tilde{\psi}|^2,$$

$$\mathcal{H}_{\text{int}} = \frac{1}{2}(U_0 - U_1)(\tilde{\psi}^* \cdot \tilde{\psi})^2 + \frac{1}{2}U_1 |\tilde{\psi}^* \times \tilde{\psi}|^2.$$

$m_l$ is longitudinal or transverse-magnetic (TM) mass,
$E_t$ is transverse or transverse-electric (TE) mass.
Consider only the kinetic energy terms (in circular basis)

\[ i \psi_{+1} = -\frac{1}{2m^*} \left[ \Delta \psi_{+1} + 4\gamma \frac{\partial^2}{\partial z^2} \psi_{-1} \right] + \ldots, \]

\[ i \psi_{-1} = -\frac{1}{2m^*} \left[ \Delta \psi_{-1} + 4\gamma \frac{\partial^2}{\partial z^2} \psi_{+1} \right] + \ldots, \]

\[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{1}{m^*} = \frac{1}{2} \left( \frac{1}{m_l} + \frac{1}{m_t} \right), \quad \gamma = \frac{m_t - m_l}{m_t + m_l}. \]

There are ±2 moment transfer between left and right components.

So, one cannot have solution like \( \psi_{+1} \propto e^{i\phi} \) and \( \psi_{-1} \propto \text{const}(\phi) \), because \( (\partial^2/\partial z^2)\psi_{-1} \propto e^{-2i\phi} \).

\[ \text{Do half-vortices exist?} \]
Elastic energy in presence of TE-TM splitting

In-plane electric field defines \( \mathbf{n} = \{\cos \eta, \sin \eta\} \). The elastic energy is

\[
H_{el} = \frac{1}{2} \int d^2 r \left( \rho_l \left\{ (\hat{n} \cdot \nabla \theta)^2 + [\hat{n} \times \nabla \eta]^2 \right\} + \rho_t \left\{ (\hat{n} \cdot \nabla \eta)^2 + [\hat{n} \times \nabla \theta]^2 \right\} \right),
\]

where \( \rho_l = \hbar^2 n/m_l \) and \( \rho_t = \hbar^2 n/m_t \).

Nonlinear field equations. Not like \( \Delta \theta = 0 \) and \( \Delta \eta = 0 \), with solutions \( \theta \propto \phi \) and \( \eta \propto \phi \) as before.

One needs to find the correct boundary conditions, i.e., \( \theta(\phi) \) and \( \eta(\phi) \) for large distances.
Asymptotic behavior

At large distances:

\[ \psi_{\pm 1}(r \gg a, \phi) = \sqrt{\frac{n}{2}} e^{i[\theta(\phi) + \eta(\phi)\phi].} \]

Vortex minimizes the energy for specific topological sector

\[ \eta(\phi + 2\pi) - \eta(\phi) = 2\pi k, \quad \theta(\phi + 2\pi) - \theta(\phi) = 2\pi m. \]

The vortex energy is \( E_{\text{vor}} = E_c + E_s \ln(R/a) \) and

\[ E_s = \frac{\hbar^2 n}{2m*} \int_0^{2\pi} \left\{ (1 + \gamma \cos(2u))(1 + u')^2 + [1 - \gamma \cos(2u)]\theta' \right\} d\phi, \]

where the prime denotes the derivative over \( \phi \) and

\[ u(\phi) = \eta(\phi) - \phi. \]
Boundary conditions

By variation we obtain

\[ [1 − \gamma \cos(2u)] \theta'' + 2\gamma \sin(2u)u'\theta' = 0, \]
\[ [1 + \gamma \cos(2u)] u'' + \gamma \sin(2u) (1 − u'^2 − \theta'^2) = 0, \]
\[ u(0) = 0, \quad \theta(0) = 0, \quad u(2\pi) = 2(\kappa − 1)\pi, \quad \theta(2\pi) = 2m\pi. \]

Simple particular vortices,

(i) Hedgehog vortices. These are \((1, m)\)-vortices having \(\theta = m\phi\) and \(u \equiv 0\), so that \(\eta = \phi\). In particular, monopole-like solution \((1, 0)\).

(ii) Double-quantized polarization vortex \((2, 0)\). In this special case \(\theta \equiv 0\), but \(u = \phi\), resulting in \(\eta = 2\phi\).
Nonlinear behavior of angles

\[ \gamma = -0.4 \text{ (thin lines)} \text{ and } \gamma = -0.9 \text{ (thick lines).} \]

The \( \left( \frac{1}{2}, \frac{1}{2} \right) \) half-vortex (a), the \( \left( -\frac{1}{2}, \frac{1}{2} \right) \) half-vortex (b).

The \( (-1, 0) \) hyperbolic polarization vortex (c), and the \( (0, 1) \) phase vortex (d).
Nonlinearity of angles (qualitatively)

Using the effective masses for the phase $m_\theta$ and for the polarization $m_\eta$,

\[
\frac{1}{m_\theta} = \frac{\cos^2 u}{m_t} + \frac{\sin^2 u}{m_l}, \quad \frac{1}{m_\eta} = \frac{\sin^2 u}{m_t} + \frac{\cos^2 u}{m_l},
\]

where $u(\phi) = \eta(\phi) - \phi$. The energy prefactor functional is

\[
E'_s = \frac{\hbar^2 n}{2} \int_0^{2\pi} \left\{ \frac{\eta'^2}{m_\eta} + \frac{\theta'^2}{m_\theta} \right\} d\phi.
\]

There are sectors where $m_\theta \approx m_l$ and $m_\eta \approx m_l$, and there are sectors where $m_\theta \approx m_t$ and $m_\eta \approx m_l$.

The main change of $\theta$ and $\eta$ happens in the sectors where the “angle” mass is heavy.
Warped half-vortices (numerical solutions)

Lemons, $k = +\frac{1}{2}$

(Monstars are not realized)

Stars, $k = -\frac{1}{2}$

$\gamma = -0.5 \quad \gamma = 0.5$
Long-range half-vortex interactions

Without TE-TM splitting.
Right and left half-vortices do not interact.
Independent proliferation of \((+\frac{1}{2}, +\frac{1}{2})\)\(-(-\frac{1}{2}, -\frac{1}{2})\) and \((\frac{1}{2}, -\frac{1}{2})\)\(-(-\frac{1}{2}, +\frac{1}{2})\).
Interaction of within each pair is \(V(r) = (1/2)E_0 \ln(r/a)\).
Two decoupled BKT transitions with \(T_c = (1/4)E_0\), where \(E_0 = \pi \hbar^2 n/m^*\).

With TE-TM splitting, \(\gamma = (m_t - m_l)/(m_t + m_l)\).
The interactions between left and right half-vortices:

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<td>((+\frac{1}{2}, +\frac{1}{2})) and ((+\frac{1}{2}, -\frac{1}{2}))</td>
<td>(-\gamma E_0 \ln(r/a))</td>
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<td>((-\frac{1}{2}, +\frac{1}{2})) and ((-\frac{1}{2}, -\frac{1}{2}))</td>
<td>(-(7/32)\gamma^2 E_0 \ln(r/a))</td>
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<td>((+\frac{1}{2}, \pm \frac{1}{2})) and ((-\frac{1}{2}, \pm \frac{1}{2}))</td>
<td>((1/8)\gamma^2 E_0 \ln(r/a))</td>
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</table>
Coupling of half-vortex cores

For attraction of polaritons with opposite spin, \( \alpha_2 = U_0 - 2U_1 < 0 \), one has the attraction of half-vortex cores.
Coupling of half-vortex cores

\[ \begin{align*}
\mathbf{r} \rightarrow (\pm \frac{1}{2}, \pm \frac{1}{2}) \quad &\text{and} \quad (-\frac{1}{2}, \pm \frac{1}{2}), &\text{stable.} \\
(-1, 0) \rightarrow (-\frac{1}{2}, +\frac{1}{2}) \quad &\text{and} \quad (-\frac{1}{2}, -\frac{1}{2}), &\text{metastable.} \\
(+1, 0) \rightarrow (+\frac{1}{2}, +\frac{1}{2}) \quad &\text{and} \quad (+\frac{1}{2}, -\frac{1}{2}), &\gamma > 0 : \text{unstable}, \quad \gamma < 0 : \text{stable.}
\end{align*} \]
Energies of warped vortices

Vortex energy $E_{\text{vor}} = E_c + E_s \ln(R/a)$, and $E_0 = \pi \hbar^2 n / m^*$. 
BKT transition temperature

The energy of a vortex $E_{\text{vor}} = E_c + E_s \ln(R/a)$.
The free energy [J. M. Kosterlitz and D. J. Thouless (1973); J. M. Kosterlitz (1974)]

$$F = E_s \ln(R/a) - TS = E_s \ln(R/a) - T \ln(R^2/a^2) = (E_s - 2T) \ln(R/a),$$

so that single vortices appear and destroy the order at $T_c = \frac{1}{2} E_s$.

Four half-vortices:

$$F = 2(E_s^{\text{star}} + E_s^{\text{lemon}}) \ln(R/a) - 4T \ln(R^2/a^2), \quad T_c = \frac{1}{4}(E_s^{\text{star}} + E_s^{\text{lemon}}).$$

Vortex molecules $(- \frac{1}{2}, \frac{1}{2}) - (1,0) - (- \frac{1}{2}, - \frac{1}{2})$:

$$F = (2E_s^{\text{star}} + E_s^{(1,0)}) \ln(R/a) - 3T \ln(R^2/a^2), \quad T_c = \frac{1}{6}(2E_s^{\text{star}} + E_s^{(1,0)}).$$

Crossover at $E_s^{(1,0)} = \frac{1}{2} \left(3E_s^{\text{lemon}} - E_s^{\text{star}}\right)$.
Vortex energy $E_{\text{vor}} = E_c + E_s \ln(R/a)$, and $E_0 = \pi \hbar^2 n/m^*$. 

For $m_t \gg m_t$, i.e, for $\gamma \to -1$, the phase transition is defined by proliferation of vortex molecules $(-\frac{1}{2}, \frac{1}{2}) - (1, 0) - (-\frac{1}{2}, -\frac{1}{2})$. 
Observation of vortices

For one-component condensate one observes the interference pattern of two beams emitted by the same condensate. One with vortex and the other without but inclined (plane wave):

$$\left| f(r)e^{i\phi} + e^{i\kappa y}\right|^2$$

For polarized condensate one studies the interference patterns in both circular polarizations.

**HQV:** fork in one circular polarization and regular fringes in the other.
Observation of half-vortices

A. x real space (µm)

B. x real space (µm)

C. x real space (µm)

D. x real space (µm)

Close pair of \((-1/2, +1/2)\) (in red circle) and \((+1/2, +1/2)\) (in blue box). These HQV form pure phase vortex \((0, +1)\) when placed together. Their close position is an indication of weak polarization pinning.

Observation of hyperbolic spin vortex (-1,0)

Observation of hyperbolic spin vortex (-1,0)

The spin-vortex \((-1,0)\) is metastable: \((-1,0) \rightarrow (-\frac{1}{2}, \frac{1}{2}) + (-\frac{1}{2}, -\frac{1}{2})\).

Acknowledgments and References

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- M. Toledo-Solano (México)
- F. Manni, Y. Léger, K. Lagoudakis, B. Deveaud-Plédran (Lausanne, Swiss)

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